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THE PHYSICAL INTERPRETATION OF THE SPINOR CONDITIONS FOR THE WAVE EQUATION

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Abstract

The nonrelativistic equation may be derived from generalized commutation relations with a matrix form of the momentum operator. The four-component spinor may be reduced to two components by a Majorana or anti-Majorana condition, which produce different coefficients of the Laplacian. The condition on the Hermitian conjugate of a Majorana fermion field is the equivalent of the anti-Majorana spinor, and a theoretical explanation is given for the essentially vanishing positron current in system consisting of negative-energy electron states. The mass of the electron would result from its representation as a predominantly Majorana spinor. The transformations from Weyl spinors to Majorana spinors can be mapped isomorphically to a subset of an $SL(2;\mathbb{R})$ subgroup of $SL(2;\mathbb{C})$, which reflects the infinitesimal mass of the neutrino. The neutrino mass term can be transformed into another spinor bilinear through this transformation. After a transition to a quantum state with the coefficient derived from this expansion, an iteration of the algorithm yields a final state with after a finite sequence. The identification of these states in each fermion generation with the leptons provides a theoretical explanation of the Koide relation for leptons and neutrinos and a prediction for m_{ν_e}, m_{ν_μ} and m_{ν_τ}. A generalized wave equation is deduced from the Hamilton-Jacobi equation for the WKB wavefunction and the path integral for a new nonrelativistic action.

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1. **Introduction**

The commutation relations of quantum mechanics can be given a functional form by equating the momentum operator with $-i\hbar\frac{d}{dx}$. The commutator of the position and momentum variables then would be $[x, p] = i\hbar$. By contrast, the Poisson bracket of these phase space coordinates in classical mechanics are $\{x, p\}_{P.B.} = 1$, and there exists no classical limit of the quantum commutator without introducing a real term into the relations.

The condition of Hermiticity of the observables then requires that the real term is not a numerical multiple of $\frac{\partial}{\partial x}$. Instead, a matrix is added to the generalized momentum. In three dimensions, it would be $\tilde{p}_i = (\kappa_0\gamma^i - i\hbar)\frac{\partial}{\partial x_i}$ [2]. The adjoint equals $\tilde{p}_i^\dagger = (\kappa_0(\gamma^i)^\dagger + i\hbar)$

$\partial x_i \sim \kappa_0\gamma^i\frac{\partial}{\partial x^i} - i\hbar\frac{\partial}{\partial x_i}$. The equation for the energy

$$\frac{p^2}{2m} + V(x) = E \tag{1.1}$$

now becomes

$$\frac{1}{2m}\left(-\kappa_0^2\mathbb{I} - \hbar^2\right)\nabla^2\psi - i\frac{\kappa_0\hbar}{m}\gamma^i\nabla^2\psi + V\psi = E\psi, \tag{1.2}$$

which represents a coupled set of equations for the components $\psi_1, , \psi_4$.

The equations may be simplified by imposing a condition on the spinor of the form $\bar{\psi} = \pm\psi^T C$, where C is the charge conjugation matrix. When $\bar{\psi} = -\psi^T C$, and the components are real, the set of coupled equations reduces to the Schrödinger equation in one dimension

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} + V(y)\psi = E\psi \tag{1.3}$$

when κ_0 is set equal to $\hbar$ after setting the derivatives with respect to the other coordinates equal to zero. However, if $\bar{\psi} = \psi^T C$ and the components are real, the nonrelativistic wave equation is altered to

$$-\frac{3\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} + V(x)\psi = E\psi. \tag{1.4}$$

The effect of the new coefficient is generally a multiplicative factor in the energy levels.

The consequences of these spinor conditions will be investigated. The fermions in four-dimensional quantum field theories usually satisfy the relations $\bar{\psi} = \psi^\dagger\gamma = \psi^T C$. These actions generally are Lorentz invariant, and a generalization to the nonrelativistic equation for scalar and spin-$\frac{1}{2}$ fields is necessary. The modified equations have been developed [1][3] and a study of the Lagrangian field theories then would follow. The anti-Majorana condition

$\bar{\psi} = -\psi^T C$ also shall be examined with respect to its effect on transformations of spinor bilinears.

The vanishing of a set of bilinears for Majorana and anti-Majorana spinors is developed in a study of the positron current. The Hermitian adjoint of a Majorana fermion field is found to be compatible with the equivalent of an anti-Majorana spinor condition. The spinor bilinear representing the current constructed from the Hermitian adjoint then vanishes for these fermions. A theoretical explanation then may be given for the nearly vanishing positron current in the system of negative energy electron states.

The electron would be a predominantly Majorana spinor field, which causes the observation of particles with the effective mass reduced by a factor of 3 in cosmic rays. The neutrino is essentially a chiral spinor, which is a nearly massless field. Both the electron and the neutrino occur in an $SU(2)$ isospin doublet in the theory of the electroweak interactions. A connection between the mass terms of the lepton and the neutrino is established by demonstrating the existence of a transformation between the Weyl spinor and the Majorana or anti-Majorana spinors. Both mass terms require a combination of different types of spinors to be nonvanishing. The neutrino spinor bilinear may be expanded to give two bilinears consisting of Majorana and anti-Majorana spinors. The first vanishes, while the second has a coefficient that is increased by a factor of 2, After a transition to a different set of four-component spinors that are embedded in an eight-component spinor similarly to the chiral spinors, an iteration of the transformation yields a sequence of scalar bilinears of quantum states. The sequence continues until a finite quantum state, which has a coefficient representing the mass of this state. The mass coincides with that of the electron in the first fermion generation. The proportionality of the mass of the lepton and the neutrino is demonstrated to be valid in the second and third fermion generations by the Koide relation. The masses of these neutrinos are predicted from the proportionality and the masses of the muon and tau lepton.

The Schrödinger equation may be deduced from a Hamilton-Jacobi equation and also the path integral. The first method consists of an equality between the wavefunction and the exponential of the WKB phase factor. The second follows from a definition of the expansion of the path integral over an infinitesimal time interval. A generalized Lagrangian in the path integral results in a modified Schrödinger equation.

2. The Effect of the Spinor Conditions in Quantum Field Theory

Beginning with the nonrelativistic matrix wave equation in component form

$$(2.1)\quad -\frac{(\kappa_0^2+\hbar^2)}{2m}\nabla^2\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

$$(2.2)\qquad -\frac{i\kappa_0\hbar}{m}\begin{pmatrix}0 & 0 & \partial_3^2 & \partial_1^2-i\partial_2^2\\ 0 & 0 & \partial_1^2+i\partial_2^2 & -\partial_3^2\\ -\partial_3^2 & -\partial_1^2+i\partial_2^2 & 0 & 0\\ -\partial_1^2-i\partial_2^2 & \partial_3^2 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

$$+V(x)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=E\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_2\\ \psi_3\\ \psi_3\\ \psi_4\end{pmatrix}$$

with $\kappa_0=\hbar$,

$$(2.3)\qquad \begin{aligned}
&-\frac{\hbar^2}{m}\nabla^2\psi_1-i\frac{\hbar^2}{2m}\partial_3^2\psi_3-i\frac{\hbar^2}{2m}\partial_1^2\psi_4-\frac{\hbar^2}{2m}\partial_2^2\psi_4+V(x)\psi_1=E\psi_1\\
&-\frac{\hbar^2}{m}\nabla^2\psi_2-i\frac{\hbar^2}{2m}\partial_1^2\psi_3+\frac{\hbar^2}{2m}\partial_2^2\psi_3+i\frac{\hbar^2}{2m}\partial_3^2\psi_4+V(x)\psi_4=E\psi_4\\
&-\frac{\hbar^2}{m}\nabla^2\psi_3+i\frac{\hbar^2}{2m}\partial_3^2\psi_1+i\frac{\hbar^2}{2m}\partial_1^2\psi_2+\frac{\hbar^2}{2m}\partial_2^2\psi_2+V(x)\psi_3=E\psi_3\\
&-\frac{\hbar^2}{m}\nabla^2\psi_4+i\frac{\hbar^2}{2m}\partial_1^2\psi_1-\frac{\hbar^2}{2m}\partial_2^2\psi_1-i\frac{\hbar^2}{2m}\partial_3^2\psi_2+V(x)\psi_4=E\psi_4.
\end{aligned}$$

The spinor now will be required to satisfy the condition $\bar{\psi} = \psi^T C$, where $C = \begin{pmatrix} 0 \\ -i\sigma^2 \\ -i\sigma^2 \\ 0 \end{pmatrix}$ is the charge conjugation matrix in the Dirac representaion. Then $\psi_1^* = \psi_4$ and $\psi_2^* = -\psi_3$ and

$$\begin{aligned} -\frac{3\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_1 &= E\psi_1 \\ -\frac{3\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_2 &= E\psi_2. \end{aligned} \tag{2.4}$$

The anti-Majorana condition $\bar{\psi} = -\psi^T C$ has the solutions $\psi_1^* = -\psi_4$ and $\psi_2^* = \psi_3$. The set of coupled differential equations [2] then is

$$\begin{aligned} -\frac{\hbar^2}{m}\nabla^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2 + i\frac{\hbar^2}{2m}\partial_1^2\psi_1^* + \frac{\hbar^2}{2m}\partial_2^2\psi_1^* + V(x)\psi_1 &= E\psi_1 \\ -\frac{\hbar^2}{m}\nabla^2\psi_2 - i\frac{\hbar^2}{2m}\partial_1^2\psi_2^* + \frac{\hbar^2}{2m}\partial_2^2\psi_2^* - i\frac{\hbar^2}{2m}\partial_3^2\psi_1^* + V(x)\psi_2 &= E\psi_2 \\ -\frac{\hbar^2}{m}\nabla^2\psi_2^* + i\frac{\hbar^2}{2m}\partial_3^2\psi_1 + i\frac{\hbar^2}{2m}\partial_1^2\psi_2 + \frac{\hbar^2}{2m}\partial_2^2\psi_2 + V(x)\psi_2^* &= E\psi_2^* \\ \frac{\hbar^2}{m}\nabla^2\psi_1^* + i\frac{\hbar^2}{2m}\partial_1^2\psi_1 - \frac{\hbar^2}{2m}\partial_2^2\psi_1 - i\frac{\hbar^2}{2m}\partial_3^2\psi_2 - V(x)\psi_1^* &= E\psi_1^*. \end{aligned} \tag{2.5}$$

The fourth equation is the negative of the conjugate of the first, and the third equation is the conjugate of the second. For a real spinor, with $\partial_1\psi = \partial_3\psi = 0$,

$$\begin{aligned} -\frac{\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_1 &= E\psi_1 \\ -\frac{\hbar^2}{2m}\partial_2^2\psi_1 + V(x)\psi_2 &= E\psi_2. \end{aligned} \tag{2.6}$$

which is the Schrödinger equation in one dimension if the potential is a function only of x^2.

The generalization of the momentum operator then may be extended to relativistic scalar fields and fermions. the analogue of the Klein-Gordon equation [3] is

$$-3\frac{\partial^2\phi}{\partial t^2} - 5\nabla^2\phi + m^2\phi = 0 \tag{2.7}$$

and the square root [1] is given by

$$(\sqrt{3}\gamma^0\partial_0 - \sqrt{5}\gamma^i\partial_i - m)\psi = 0. \tag{2.8}$$

It may be recalled that the relativistic formula

$$E = \sqrt{c^2|\vec{p}|^2 + m^2c^4} \tag{2.9}$$

can be expanded as

$$E = mc^2 + \frac{|\vec{p}|^2}{2m} - \frac{|\vec{p}|^4}{8m^3c^2} + \dots . \tag{2.10}$$

Incorporating the rest mass energy mc^2 into $V(x)$, the classical approximation $E_{tot} = \frac{|\vec{p}|^2}{2m} + V(x)$ would be valid to $O\left(\frac{v^4}{c^4}\right)$ when $v \ll c$.

Therefore, the nonrelativistic equation must be recovered in this limit. It is consistent, therefore, to consider the spinor conditions in the relativistic field theories. First, when $\bar{\psi} = \psi^T C$, the equation of motion for the fermion is given by Eq.(2.9), derived from from the modified Klein-Gordon equation (2.8), which is equivalent to the energy formula

$$\tilde{E}^2 \sim \frac{5}{3}c^2|\vec{p}_\ell|^2 + \frac{m^2c^4}{3} \tag{2.11}$$

where the energy is defined with respect to the Euclidean coordinate τ, with $t = i\tau$. The momentum vector $p^\mu = (p^0\vec{p})$ is a Lorentz four-vector, and yet the derivatives in the classical mechanics may defined with respect to a time coordinate that is independent of the spatial coordinates. Furthermore, with the magnitude of momentum $|\vec{p}|^2 = m\left|\frac{d\vec{x}}{dt}\right|$ $= m\left|-i\frac{d\vec{x}}{d\tau}\right|^2 = m\left|\frac{d\vec{x}}{d\tau}\right|^2$, and the entire formula (2.12) can be interpreted with a single time variable. It is clearly a modification of the standard relation between energy and momentum which occurs also in the formula obtained from the nonrelativistic wave equation (2.5) with a coefficient of 3.

The fermion action giving rise to Eq.(2.9) is

$$S_{ferm.} = \int d^4x\bar{\psi}\left[\sqrt{3}\gamma^0\partial_0 - \sqrt{5}i\gamma^i\partial_i - m\right]\psi. \tag{2.12}$$

and the quantum propagator [1] equals

$$S_{F\alpha\beta} = 25i\frac{((1+i\gamma_5)(\gamma\cdot p + m))_{\alpha\beta}}{5E^2 + 3|\vec{p}|^2 + m^2}. \tag{2.13}$$

in units with $\hbar = 1$. The scattering cross sections for various reactions near the classical scales then may be computed through the diagrammatic expansion of the S-matrix, Now suppose that the conjugate of the fermion field is $\bar{\psi} = -\psi^T C$. The Schrödinger equation is recovered in the nonrelativistic limit, and the quantization of the relativistic action then could be developed with the standard energy formula without matrix operators. The Klein-Gordon and Dirac equations would be derived. Both the action and the perturbation series would be similar to the standard expressions except that the spinor bilinears would be defined to be $\bar{\psi}\gamma^{\mu_1}...\gamma^{\mu_r}\chi = -\psi^T C\gamma^{\mu_1}...\gamma^{\mu_r}\chi$.

Consider spinor bilinears $\bar{\psi}$ and $\bar{\psi}\gamma^\mu\psi$, where the Dirac and Majorana conjugates are equal, $\bar{\psi} = \psi^\dagger\gamma^0 = \psi^T C$. While

For an anti-Majorana spinor

$$\begin{aligned} \bar{\psi}\psi &= -\psi^T C\psi = -(\psi^T C\psi)^T = -\psi^T C^T\psi \\ &= \psi^T C\psi = -(-\psi^T C)\psi - -\bar{\psi}\psi \end{aligned} \tag{2.14}$$

and $\bar{\psi}\psi$ must equal zero, while

$$\begin{aligned} \bar{\psi}\gamma^\mu\psi &= -\psi^T C\gamma^\mu\psi = -\psi^T(-\gamma^\mu)^T C\psi \\ &= \psi^T(\gamma^\mu)^T C\psi = (\psi^T(\gamma^\mu)^T C\psi)^T = \psi^T C^T\gamma_\mu\psi \\ &= -\psi^T C\gamma^\mu\psi \\ &= \bar{\psi}\gamma^\mu\psi. \end{aligned} \tag{2.15}$$

Again, the current density can be nonvanishing. The other bilinears are

$$\begin{aligned} \bar{\psi}\gamma_5\psi &= -\psi^T C\gamma_5\psi = -\psi^T\gamma_5^T C\psi \\ &= -(\psi^T\gamma_5^T C\psi)^T = -\psi^T C^T\gamma_5\psi \\ &= \psi^T C\gamma_5\psi - = -\bar{\psi}\gamma_5\psi \end{aligned} \tag{2.16}$$

$$\begin{aligned} \bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi &= -\psi^T C\gamma^{[\mu}\gamma^{\nu]}\psi = -\psi^T(-\gamma^{[\mu T})(-\gamma^{\nu]T})C\psi \\ &= -\psi^T\gamma^{[\nu\ T}\gamma^{\mu]T}C\psi = -(\psi^T\gamma^{[\nu\ T}\gamma^{\mu]T}C\psi)^T \\ &= -\psi^T C^T\gamma^{[\nu}\gamma^{\mu]}\psi = -\psi^T(-C)(-\gamma^{[\mu}\gamma^{\nu]})\psi \\ &= -\psi^T C\gamma^{[\mu}\gamma^{\nu]}\psi = \bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}(2.18)\qquad \bar{\psi}\gamma^\mu\gamma_5\psi &= -\bar{\psi}^T C\gamma^\mu\gamma_6\psi = -\psi^T(-\gamma^{\mu\ T}C\gamma_5\psi \\ &= \psi^T\gamma^{\mu\ T}\gamma_5^T C\psi = (\psi^T\gamma^{\mu\ T}\gamma_5^T C\psi)^T = \psi^T C^T\gamma_5\gamma^\mu\psi \\ &= -\psi^T C\gamma_5\gamma^\mu\psi = \psi^T C\gamma^\mu\gamma_5\psi \\ &= -\bar{\psi}\gamma^\mu\gamma_5\psi.\end{aligned}$$

Consequently, $\bar{\psi}\gamma_5\psi$ and $\bar{\psi}\gamma^\mu\gamma_5\psi$ would equal zero and there are no Lagrangian field theories consisting of a fermion field satisfying the anti-Majorana spinor condition with axial currents.

3. Particle and Antiparticle Currents

The expansion of a fermion field psi is

$$(3.1)\qquad \psi(x) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2p^0}\sum_s\,[a_s^\dagger(\vec{p})e^{-ip\cdot x}u_s(p) + b_s^\dagger(\vec{p})e^{ip\cdot x}v_s(p)]$$

where $a_s(\vec{p})$ is annihilation operator of a particle state, $b_s^\dagger(\vec{p})$ is a creation operator of an anti-particle state of momentum $\vec{p}$ and helicity s, and $u_s(\vec{p})$ and $v_s(\vec{p})$ are two-component spinors normalized by $u_r^\dagger(\vec{p})u_s(\vec{p}) = v_r^\dagger(\vec{p})v_s(\vec{p}) = 2E_p\delta_{rs}$. Both sets of terms represent the equivalent of the annilhilation of a single fermion state, since the creation of the antiparticle is interpreted to be the removal of an electron. The hermitian adjoint of this field is

$$(3.2)\qquad \psi^\dagger(x) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2p^0}\sum_s\,[a_s^\dagger(\vec{p})e^{ip\cdot x}]u_s^\dagger(\vec{p}) + b_s(\vec{p})e^{-ip\cdot x}v_s^\dagger(\vec{p})]$$

there describes the expansion of the antiparticle field. Nevertheless, the formulation of these integrals is consistent with ψ annihilating particle states and $\psi^\dagger$ creating particle states. Suppose that $\psi(x)$ is a Majorana spinor. Then

$$(3.3)\qquad \overline{\bar{\psi}^\dagger} = (\psi^T C)^\dagger = (C^T)^*(\psi^{dag})^T = -C(\psi^\dagger)^T$$

which is the equivalent of the anti-Majorana condition. From §2, $\bar{\chi}\gamma^\mu\chi$ is not required to be zero when χ is an anti-Majorana spinor. However, given that ψ is a Majorana spinor and $\psi^\dagger$ is an anti-Majorana spinor,

$$\begin{aligned}(3.4)\qquad \psi^\dagger &= \bar{\psi}\gamma^0 = \psi^T C\gamma^0 \\ \bar{\psi}^\dagger &= (\psi^\dagger\gamma^0)^\dagger = \gamma^0\psi\end{aligned}$$

where C is the charge conjugation matrix with $C^{-1} = C^T = -C$, $C^* = C$, and

$$(3.5) \qquad \begin{aligned} \psi_A^\dagger \gamma^\mu \overline{\psi_A^\dagger} &= -\psi_A^T C \gamma^0 \gamma^\mu \gamma^0 \psi_M = -\psi_A^T C (\gamma^\mu)^\dagger \psi_M \\ &= -\psi_M^T (\gamma^\mu)^* C^T \psi_A = -(\psi^\dagger \gamma^0 C^{-1})_M (\gamma^\mu)^* C^T \gamma^0 \bar{\psi}_A^\dagger \\ &= \psi_A^\dagger \gamma^0 C^{-1} (\gamma^\mu)^* C \gamma^0 \bar{\psi}_A^\dagger = \psi_A^\dagger \gamma^0 (-\gamma^\mu)^\dagger \gamma^0 \bar{\psi}_A^\dagger \\ &= -\psi_M^\dagger \gamma^\mu \overline{\psi_A^\dagger}, \end{aligned}$$

wherer ψ_M is the Majorana fermion and $\psi_A^\dagger$ is the anti-Majorana anti-fermion. The difference between this bilinear and Eq.(2.20) is the role of ψ as a Majorana spinor, which yields the identity $\psi^\dagger = \psi^T \gamma^0 C^{-1}$. Consequently, the current $\psi^\dagger \gamma^\mu \overline{\psi^\dagger}$ vanishes.

The physical effect of this vanishing current may be related to the filling of the negative energy electron states [5]. The zero-energy state, with no positive energy electrons, generally would be regarded as inert without any movement or interactions. The scattering off a system in this zero-energy state releases an electron from one of the negative energy states, which cannot be reached from positive energies by the exclusion principle, and creates a positron. It is evident that the energy is imparted to the electron externally with positron remaining nearly static internally within the initially zero-energy system. Consequently, the electrons may generate a current, while there may be essentially no positron current.

Positrons have been identified with electrons moving backwards in time [7]. If an electron wavefunction is represented as $u_s(\vec{p})e^{-ip\cdot x} = u_s(\vec{p})e^{-iE_pt+i\vec{p}\cdot x}$, the reversal of time would transform the positive frequency mode to $e^{iE_pt+i\vec{p}\cdot\vec{x}}$ or equivalently $e^{-i(-E_p)t+i\vec{p}\cdot\vec{x}}$, which has negative energy. Therefore, the positron state can be viewed alternatively as a negative energy electron state.

The zero-energy state then consists of a continuum of positrons without only negligible motion. The positron current then would be almost zero. By contrast, it is possible for a nonvanishing electron current externally to this state. The vanishing of the current for $\psi^\dagger$ then verifies the dynamics of this physical model of the electron as a predominantly Majorana spinor field.

A general fermion field ψ can be written as a linear combination of a Majorana spinor ψ_1 and an anti-Majorana spinor ψ_1 since

$$\psi = k_1\psi_1 + k_2\psi_2 = k_1 \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ -\psi_{12}^* \\ \psi_{11}^* \end{pmatrix} + k_2 \begin{pmatrix} \psi_{21} \\ \psi_{22} \\ \psi_{22}^* \\ -\psi_{21}^* \end{pmatrix} \tag{3.6}$$

because there are eight equations for twelve unknowns $Re\ k_1$, $Im\ k_1$, $Re\ k$, $Im\ k_2$, $Re\ \psi_{11}$, $Im\ \psi_{11}$, $Re\ \psi_{12}$, $Im\ \psi_{12}$, $Re\ \psi_{21}$, $Im\ \psi_{21}$, $Re\ \psi_{22}$ and $Im\ \psi_{22}$. It follows that

$$\bar{\psi}\gamma^\mu\psi = |k_1|^2\bar{\psi}_1\gamma^\mu\psi_1 + k_1^*k_2\bar{\psi}_1\gamma^\mu\psi_2 + k_1k_2^*\bar{\psi}_2\gamma^\mu\psi_2k_1k_2^* + |k_2|^2\bar{\psi}_2\gamma^\mu\psi_2. \tag{3.7}$$

Since

$$\begin{aligned} \bar{\psi}_1\gamma^\mu\psi_2 &= \psi_1^T C\gamma^\mu\psi_2 = \psi_1^T(-\gamma^\mu)^T C\psi \\ &= (-\psi_1^T\gamma^{\mu\ T}C\psi)^T = -\psi_1^T C^T\gamma^\mu\psi_1 \\ &= \psi_2^T C\gamma^\mu\psi_1 = -\bar{\psi}_2\gamma^\mu\psi_1, \end{aligned} \tag{3.8}$$

$$\bar{\psi}\gamma^\mu\psi = |k_1|^2\bar{\psi}_1\gamma^\mu\psi_1 + |k_2|^2\bar{\psi}_2\gamma^\mu\psi_2 + (k_1^*k_2 - k_1k_2^*)\bar{\psi}_1\gamma^\mu\psi_2 \tag{3.9}$$

Currents consist of coupling of Majorana spinors or anti-Majorana spinors only when $arg\ k_1 = arg\ k_2$ or $arg\ k_1 = arg\ k_2 + \pi$. Furthermore,

$$\begin{aligned} (\bar{\psi}_1\gamma^\mu\psi_2)^* &= (\psi_1^\dagger\gamma^0\gamma^\mu\psi_2)^* = \psi_1^T\gamma^0\gamma^{\mu\ *}\psi_2^* = (\psi_1^T\gamma^0(\gamma^{\mu\ *}\psi_2)^T \\ &= \psi_1^T(\gamma^\mu)^\dagger\gamma^\mu\psi_1 = \bar{\psi}_2\gamma^\mu\psi_1 \end{aligned} \tag{3.10}$$

Therefore, $\bar{\psi}_1\gamma^\mu\psi_2$ is imaginary and

$$k_1^*k_2\bar{\psi}_1\gamma^\mu\psi_2 + k_1k_2^*\bar{\psi}_2\gamma^\mu\psi_1 = 2\ Re\ [k_1^*k_2\bar{\psi}_1\gamma^\mu\psi_2] \tag{3.11}$$

is always real. It again vanishes when $arg\ k_1 = arg\ k_2$ and $arg\ k_1 = arg\ k_2 + \pi$.

The mass term in the Lagrangian equals

$$m\bar{\psi}\psi = m(|k_1|^2\bar{\psi}_1\psi_1 + |k_2|^2 + k_1^*k_2\bar{\psi}_1\psi_2 + k_1k_2^*\bar{\psi}_2\psi_1. \tag{3.12}$$

By §2, $\bar{\psi}_1\psi_1 = \bar{\psi}_2\psi_2 = 0$, while

$$\bar{\psi}_1\psi_2 = \psi_1^T C\psi_2 = (\psi_1^T C\psi_2)^T = \psi_2^T C^T\psi_1 = -\psi_2^T C\psi_1 = \bar{\psi}_2\psi_1. \tag{3.13}$$

and

$$(\bar{\psi}_1\psi_2)^* = \bar{\psi}_1^*\psi_2^* = (\psi_1^\dagger\gamma^9)^*(\psi_2^\dagger)^T \\ = \psi_1^T\gamma^0(\psi_2^\dagger)^T = (\psi_1^T\gamma^0(\psi_2^\dagger)^T)^T \\ = \psi_2^\dagger\gamma^0\psi_1 = \bar{\psi}_2\psi_1. \tag{3.14}$$

The mass term equals

$$(k_1^*k_2 + k_1k_2^*)\bar{\psi}_1\psi_2 = 2\ Re[k_1^*k_2\ \bar{\psi}_1\psi_2]. \tag{3.15}$$

Since $\bar{\psi}_1\psi_2$ is real, this expression is real if $arg\ k_1 = arg\ k_2$ or $arg\ k_1 = arg\ k_2 + \pi$.

The kinetic term equals

(3.16)

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi = i[|k_1|^2{}_1\gamma^\mu\partial_\mu\psi_1 + |k_2|^2\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 + k_1^*k_2\bar{\psi}_1\gamma_\mu^\partial\psi_2 + k_1k_2^*\bar{\psi}_2\gamma^\mu\partial_\mu\psi_1].$$

Since

$$i\int_M d^4x\ [k_1^*k_2\bar{\psi}_1\gamma^\mu\partial_\mu\partial_\mu\psi_2 + k_1k_2^*\bar{\psi}_2\gamma^\mu\partial_\mu\psi_1] \\ = i\int_M d^4x[k_1^*k_2\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2 - k_1k_2^*\overline{\partial_\mu\psi_1}\gamma^\mu\psi_2] \\ = i\int_M d^4x\ [k_1k_2^*\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2 - k_1k_2^*\partial_\mu(\bar{\psi}_1\gamma^\mu\psi_2) + k_1k_2^*\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2] \\ = i\int_M d^4x(k_1^*k_2 + k_1k_2^*)\bar{\psi}_1\gamma^\mu\partial_{\mu 2} - ik_1k_2^*\int_{\partial M} d^3\Sigma_\mu\bar{\psi}_1\gamma^\mu\psi_2, \tag{3.17}$$

and the boundary integral vanishes when $\psi_1,\ \psi_2 \sim \frac{1}{r^{\frac{3}{2}+\epsilon}}$, $\epsilon > 0$, under these conditions, the kinetic action equals

$$i\int_M d^4x\ [|k_1|^2\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 + |k_2|^2\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 + (k_1^*k_2 + k_1k_2^*)\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2]. \tag{3.18}$$

The complex conjugate of $i\bar{\psi}_1\gamma_\mu^\partial\psi_1$

$$(i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1)^* = -i(\psi_1^\dagger\gamma^0)^\mu{}^*\partial_\mu\psi_1^* = -i\psi_1^T\gamma^0\gamma^\mu{}^*\partial_\mu\psi_1^* \\ = -i\partial_\mu\psi_1^\dagger\gamma^{\mu\dagger}\gamma^0\psi_1 = -i\partial_\mu\psi_1^\dagger\gamma^{\gamma\mu}\psi_1 \tag{3.19}$$

can be integrated to give

$$-i\int_M d^4x\ \partial_\mu\psi_1^T\gamma^0\gamma^\mu\psi_1 = -i\int_M d^4x\ [\partial_\mu(\psi_1^\dagger\gamma^0\gamma^\mu\psi_1) - \psi_1^\dagger\gamma^0\gamma^\mu\partial_\mu\psi_1] \\ = i\int_M d^4x\ \bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 - i\int_M d^4x\ \bar{\psi}1\gamma^\mu\psi_1. \tag{3.20}$$

The boundary integral again vanishes under the same dependence on the distance near spatial infinity, and

$$\left(i\int_M d^4x\ \bar{\psi}_1\gamma^\mu\partial_\mu\psi_1\right)^* = i\int_M d^4x\ \bar{\psi}_1\gamma^\mu\partial_\mu\psi_2. \tag{3.21}$$

Similarly,

$$\left(i\int_M d^4x\ \bar{\psi}_2\gamma^\mu\partial_\mu\psi_2\right)^* = i\int d^4x\ \bar{\psi} + 2\gamma^\mu\partial_\mu\psi_2. \tag{3.22}$$

The complex conjugate of $i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2$ is

$$\begin{aligned}(i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2)^* &= -i(\psi_1^\dagger\gamma^0\gamma^\mu\partial_\mu\psi_2)^* = -i\psi^T\gamma^0(\gamma^\mu)^*\partial_\mu\psi_2^* \\ &= -i\partial\psi_2^\dagger(\gamma^\mu)^\dagger\gamma^0\psi_1 == i\partial_\mu\psi_2^\dagger(\gamma^\mu)^\dagger\gamma^\mu\psi_1 \\ &= -i\partial_\mu\psi_2^\dagger\gamma^0\gamma^\mu\psi_1.\end{aligned} \tag{3.23}$$

which is integrated to

$$\begin{aligned}-i\int_M d^4x\ \partial\bar{\psi}_2\gamma^\mu\psi_1 &= -i\int_M d^4x\ [\partial_\mu(\bar{\psi}_2^\mu\psi_1) - \bar{\psi}_2\gamma^\mu\partial_\mu\psi_1] \\ &= i\int_M d^4x\bar{\psi}_2\gamma^\mu\partial_\mu\psi_1 - i\int_{\partial M} d\Sigma_\mu\ \bar{\psi}_2\gamma^\mu\psi_1\end{aligned} \tag{3.24}$$

and

$$\left(i\int d^4x\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2\right)^* = i\int d^4x\bar{\psi}_1\gamma^\mu\partial_\mu\psi_2. \tag{3.25}$$

Therefore, the fermion action is real if $arg\ k_1 = arg\ k_2$ or $arg\ k_1 = arg\ k_2 + \pi$.

When $|k_1| \gg |k_2|$, the fermion field is predominantly a Majorana spinor, the mass is proportional to $Re(k_1^* k_2)$, and the current is nearly equal to $|k_1|^2\ i\ \int d^4x\ \bar{\psi}_1\gamma^\mu\partial_\mu\psi_1$. The current then can be large even for an infinitesimal mass. Since the electron does have a non-zero mass, $k_1(e^-) \gg k_2(e^-) > 0$. It would be a field with a mainly Majorana spinor and an infinitesimal anti-Majorana spinor component.

4. A Transformation from Weyl Spinors to Majorana Spinors

The neutrino is generally described by a left-handed Weyl spinor ν_L. This field and the electron form an $SU(2)$ doublet in the standard model. Experimental data from cosmic rays reflect the occurence of a reaction $e^- \to W^-\nu_e$, with the effective mass of the W^- boson being reduced by an effective factor of 3 [4][12]. A study of the nonrelativistic form of the wave equation for a real

Majorana spinors demonstrates that the wavefunction would be a solution to a Schrödinger equation

$$-\frac{3\hbar^2}{2m}\partial_y^2\psi + V(y)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{4.1}$$

with an effective mass reduced by a factor of 3. Consequently, the electron may be regarded as a primarily Majorana spinor.

Since $\begin{pmatrix} e \\ \nu_L \end{pmatrix}$ is an SU(2) isospin doublet, the existence of a rotation from a Weyl to a Majorana spinor may be examined. The gamma matrices will be defined in the chiral representation in the remainder of this section. With the matrix $\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$, the chirality conditions $(1-\gamma_5)\psi_L = 0$ and $(1+\gamma_5)\psi_R = 0$ have the solutions

$$\psi_R = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \psi_L = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix} \tag{4.2}$$

Consider first the transformation from ψ_R to the Majorana spinor

$$\psi_M = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_2^* \\ \psi_1^* \end{pmatrix} \equiv \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ -\psi_{21} & \psi_{22} \\ \psi_{11} & -\psi_{12} \end{pmatrix}, \tag{4.3}$$

where $\psi_1 \equiv \psi_{11} + i\psi_{12}$ and $\psi_2 = \psi_{21} + i\psi_{22}$. The matrix equation

$$A\psi_R = \psi_M, \tag{4.4}$$

has the solution

(4.5)
$$\begin{aligned} a_{11} &= 1 & a_{12} &= 0 \\ a_{21} &= 0 & a_{22} &= 1 \\ a_{31} &= -\frac{2\psi_{21}\psi_{22}}{det\ \Psi_2} & a_{32} &= \frac{\psi_{11}\psi_{22}+\psi_{12}\psi_{21}}{det\ \Psi_2} \\ a_{41} &= \frac{\psi_{11}\psi_{22}+\psi_{12}\psi_{21}}{det\ \Psi_2} & a_{42} &= -\frac{2\psi_{11}\psi_{12}}{det\ \Psi_2} \\ \Psi_{2R} &= \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}. \end{aligned}$$

The remaining elements are undetermined. The matrix can be completed to

(4.6)
$$\begin{aligned} A &= \begin{pmatrix} I_2 & 0 \\ A_2 & -I_2 \end{pmatrix} \\ A_2 &= \frac{1}{det\ \Psi_2}\begin{pmatrix} -2\psi_{21}\psi_{22} & \psi_{11}\psi_{22}+\psi_{12}\psi_{21} \\ \psi_{11}\psi_{22}+\psi_{12}\psi_{22} & -2\psi_{11}\psi_{12} \end{pmatrix}, \end{aligned}$$

with the identities

(4.7)
$$\begin{aligned} det\ A_2 &= -1 \\ A^2 &= I_4 \\ det\ A &= 1. \end{aligned}$$

This formula is consistent with an automorphism of a doublet. The matrix $\mathbb{A}$ belongs to a $\mathbb{Z}_2$ subgroup of $\mathrm{SL}(4;\mathbb{C})$. In the complex formulation,

(4.8)
$$A_2^{\mathbb{C}}\begin{pmatrix} \psi_1 \\ \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\psi_2^* \\ \\ \psi_1^* \end{pmatrix}$$

and $A_2^{\mathbb{C}}$ preserves the norm of vectors in $\mathbb{C}^2$. Since $|det\ A_2^{\mathbb{C}}| = 1$, and there is a bijective correspondence between the matrices of the form $A \in \mathrm{SL}(4;\mathbb{R})$, $A_2^{\mathbb{C}} \in \mathrm{SL}(2;\mathbb{C}) \cup \mathrm{SL}^-(2;\mathbb{C})$, where $\mathrm{SL}^-(2;\mathbb{C})$ is the set of special linear transformations in $\mathbb{C}^2$ with determinant -1. Since $A \in SL(4;\mathbb{R})$ and $det\ A_2 = -1$, $A_2 \in \mathrm{SL}^-(2;\mathbb{R}) \subset \mathrm{SL}^-(2;\mathbb{C})$. Multiplication by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives an element of $\mathrm{SL}(2;\mathbb{R}) \subset \mathrm{SL}(2;\mathbb{C})$

(4.9)
$$\hat{A}_2 = A_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{det\ \Psi_{2R}}\begin{pmatrix} \psi_{11}\psi_{22}+\psi_{12}\psi_{21} & -2\psi_{21}\psi_{22} \\ -2\psi_{11}\psi_{12} & \psi_{11}\psi_{22}+\psi_{12}\psi_{21} \end{pmatrix}.$$

There is a bijection from the set $\{A\gamma_5\}$ to $\left\{ \begin{pmatrix} I & 0 \\ \hat{A}_2 & I \end{pmatrix} \right\}$, and a subset of an $\mathrm{SL}(2;\mathbb{R})$ subgroup of $\mathrm{SL}(2;\mathbb{C})$.

A solution to the equation

$$B\psi_L = \psi_M = \begin{pmatrix} \psi_4^* \\ -\psi_3^* \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{4.10}$$

is

$$\begin{aligned} B &= \begin{pmatrix} -I_2 & B_2 \\ 0 & I_2 \end{pmatrix} \\ B_2 &= \frac{1}{det\ \Psi_{2L}} \begin{pmatrix} \psi_{41}\psi_{42} \\ -(\psi_{31}\psi_{42} + \psi_{32}\psi_{41}) \\ -(\psi_{31}\psi_{42} + \psi_{32}\psi_{41}) \\ 2\psi_{31}\psi_{41} \end{pmatrix} \\ \Psi_{2L} &= \begin{pmatrix} \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix}. \end{aligned} \tag{4.11}$$

It follows that

$$\begin{aligned} det\ B_2 &= -1 \\ B^2 &= I_4 \\ det\ B &= 1. \end{aligned} \tag{4.12}$$

There is a bijective correspondence between the set of matrices $\{-B\gamma_5\}$ in $\mathrm{SL}(4;\mathbb{R})$ and a subset of $\mathrm{SL}(2,\mathbb{R})$.

Similarly, the transformations from the chiral spinors to an anti-Majorana spinor

$$\begin{aligned} A'\psi_R &= \psi_A \\ B'\psi_L &= \psi_A \end{aligned} \tag{4.13}$$

are given by

$$\begin{aligned} A' &= \begin{pmatrix} I_2 & 0 \\ A_2' & -I_2 \end{pmatrix} \\ A_2' &= -A_2 \end{aligned} \tag{4.14}$$

and

(4.15)
$$B' = \begin{pmatrix} -I_2 & B_2' \\ 0 & I_2 \end{pmatrix}$$
$$B_2' = -B_2$$

The sets $\{A'\gamma_5\}$ and $\{-B'\gamma_5\}$ can be mapped bijectively to subsets of SL$(2;\mathbb{R})$.

The bilinears $\bar{\psi}_R\psi_R$ and $\bar{\psi}_L\psi_L$ vanish, and chiral spinors describe massless fields. Suppose that the nearly chiral spinor is $\nu = \kappa_1\nu_L + \kappa_2\nu_R$, where $\kappa_1 \gg \kappa_2$. Then

(4.16)
$$m\bar{\nu}\nu = m[\kappa_1^*\kappa_2\bar{\nu}_L\nu_R + \kappa_1\kappa_2^*\bar{\nu}_R\nu_L] = 2Re[\kappa_1^*\kappa_2\bar{\nu}_L\nu_R].$$

The initial set of solutions to Eq.(4.11) yields

(4.17)
$$\bar{\nu}_L^\dagger\gamma^0 \to \nu_M^\dagger(B^{-1})^\dagger\gamma^0 = \nu_M^\dagger B^T\gamma^0 = \bar{\nu}_M\gamma^0 B^T\gamma^0.$$

If

(4.18)
$$\nu_R \to (B')^{-1}\nu_A = B'\nu_A.$$

(4.19)
$$\bar{\nu}_L\nu_R \to \bar{\nu}_M\gamma^0 B^T\gamma^0 B'\nu_A$$

From

(4.20)
$$\gamma^0 B^T\gamma^0 B' = -I_4,$$

it follows that

(4.21)
$$\bar{\nu}_L\nu_R \to -\bar{\nu}_M\nu_A.$$

When the solutions $\{A, B, C, D\}$ are replaced by $\{A\gamma_5, -B\gamma_5, C\gamma_5, -D\gamma_5\}$,

(4.22)
$$\bar{\nu}_L \to -\nu_M^\dagger(B^{-1})^T\gamma_5\gamma^0 = -\bar{n}u_M\gamma^0(B^{-1})^T\gamma_5\gamma^0,$$

(4.23)
$$\nu_R \to -\gamma_5(B'^{\,-1})\nu_A$$

and

(4.24)
$$\bar{\nu}_L\nu_R \to= -\bar{\nu}_M\gamma^0(B^{-1})^T\gamma_5\gamma^0(-\gamma_5(B')^{-1})\nu_A = -\bar{\nu}_M\gamma^0(B^{-1})^T\gamma^0(B')^{-1}\nu_A.$$

Since

(4.25)
$$-\gamma^0(B^{-1})^T\gamma_5\gamma^0(B')^{-1} = -\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}\begin{pmatrix} -I_2 & 0 \\ -B_2 & I_2 \end{pmatrix}\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}\begin{pmatrix} -I_2 & B_2 \\ 0 & I_2 \end{pmatrix}$$
$$= \begin{pmatrix} I_2 & 2B_2 \\ 0 & I_2] \end{pmatrix},$$

$$\bar{\nu}_L \nu_R \to \bar{\nu}_M \nu_A + 2\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A. \tag{4.26}$$

It may be noted that ν_L and ν_R can be derived from two different spinors. If $\nu_L = \frac{1}{2}(1-\gamma_5)\nu$ and $\nu_R = \frac{1}{2}(1+\gamma_5)\nu$,

$$\begin{aligned} \bar{\nu}_L \nu_R &= (0\ 0\ \nu_3^*\ \nu_4^*) \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ 0 \\ 0 \end{pmatrix} \\ &= \nu_3^* \nu_1 + \nu_4^* \nu_1. \end{aligned} \tag{4.27}$$

When the spinors ν_M and ν_A are set equal to $\begin{pmatrix} \nu_1 \\ \nu_2 \\ -\nu_2^* \\ \nu_1^* \end{pmatrix}$ and $\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_2^* \\ -\nu_1^* \end{pmatrix}$, respectively, in the mass term $m\bar{\nu}_M \nu_R$ vanishes because $\bar{\nu}_M \nu_A = 0$.

A coincidence of the spinors generating ν_M and ν_A will be necessary for the reduction of any remaining bilinears. Given that

$$\begin{aligned} B_2 &= \frac{1}{det\ V_{2L}} \begin{pmatrix} 2\nu_{31}\nu_{42} & -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41}) \\ -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41} & 2\nu_{31}\nu_{42} \end{pmatrix} \\ V_{2L} &= \begin{pmatrix} \nu_{31} & \nu_{32} \\ \nu_{41} & \nu_{42} \end{pmatrix}, \end{aligned} \tag{4.28}$$

$$\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A = (\nu_{3\,4}^{\nu *}) B_2 \begin{pmatrix} \nu_3 \\ \nu_4 \end{pmatrix}. \tag{4.29}$$

Expressing the complex component of the spinor as two real components, the above matrix product can be written in the form

(4.30)
$$\begin{aligned}
&\frac{1}{det\ V_{2L}} \begin{pmatrix} \nu_{31} & \nu_{41} \\ -\nu_{32} & -\nu_{42} \end{pmatrix} \begin{pmatrix} 2\nu_{41}\nu_{42} & -(\nu_{31}\nu_{42}+\nu_{32}\nu_{41}) \\ -(\nu_{31}\nu_{42}+\nu_{32}\nu_{41}) & 2\nu_{31}\nu_{32} \end{pmatrix} \\
&\qquad = \frac{1}{det\ V_{2L}} \begin{pmatrix} \nu_{31} & \nu_{41} \\ -\nu_{32} & \nu_{42} \end{pmatrix} det(V_{2L}) \begin{pmatrix} \nu_{41} & -\nu_{42} \\ -\nu_{31} & \nu_{32} \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0 & -det\ V_{2L} \\ det\ V_{2L} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} det\ V_{2L},
\end{aligned}$$

The bilinear $\bar{\nu}_M \nu_A$ is real because it equals the complex conjugate $(\bar{\nu}_M \nu_A)^* = \bar{\nu}_A \nu_M$. By contrast,

$$\begin{aligned}
(4.31)\quad \left(\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\right)^* &= \left(\nu_M^\dagger \gamma^0 \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\right)^* \\
&= \left(\nu_M^T \gamma^0 \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A^*\right)^T \\
&= \nu_A^\dagger \begin{pmatrix} 0 & 0 \\ B_2 & 0 \end{pmatrix} \gamma^0 \nu_M = \bar{\nu}_M \gamma^0 \begin{pmatrix} 0 \\ B_2 \\ 0 \end{pmatrix} \gamma^0 \nu_M
\end{aligned}$$

Since

$$\gamma^0 \begin{pmatrix} 0 & 0 \\ B_2 & 0 \end{pmatrix} \gamma^0 = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}, \tag{4.32}$$

$$\left(\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\right)^* = \bar{\nu}_A \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_M. \tag{4.33}$$

However,

$$\begin{aligned}
(4.34)\quad \bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A &= \nu_M^T C \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A = \left(\nu_M^T C \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\right)^T \\
&= \nu_A^T \begin{pmatrix} 0 & 0 \\ B_2 & 0 \end{pmatrix} C^T \nu_M = -\nu_A^T \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} C \nu_M
\end{aligned}$$

With the matrix products

(4.35)
$$\begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & -iB^2\sigma^2 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma^2 B_2 \\ 0 & 0 \end{pmatrix},$$

(4.36)

$$-\nu_A^T \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} C\nu_M = -\nu_A^T C \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_M - \nu_A^T \begin{pmatrix} 0 & -i[B_2, \sigma_2] \\ 0 & 0 \end{pmatrix} \nu_M$$
$$= \bar{\nu}_A \begin{pmatrix} B_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \nu_M - \nu_A^T \begin{pmatrix} 0 & -i[B_2, \sigma^2] \\ 0 & 0 \end{pmatrix} \nu_M$$

Therefore, $\bar{n}u_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A$ is a complex number and

(4.37)
$$Im\Big[\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\Big] = -\frac{1}{2i}\nu_A^T \begin{pmatrix} 0 & -i[B_2, \sigma^2] \\ 0 & 0 \end{pmatrix} \nu_M$$

The commutator equals

(4.38)
$$[B_2, -i\sigma^2] = \left[B_2, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$$
$$= -\frac{1}{det\ V_{2L}} \Bigg[\begin{pmatrix} \nu_{31}\nu_{42} + \nu_{32}\nu_{41} & -2\nu_{31}\nu_{32} \\ 2\nu_{41}\nu_{42} & -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41} \end{pmatrix}$$
$$- \begin{pmatrix} -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41} & 2\nu_{31}\nu_{32} \\ -2\nu_{41}\nu_{42} & \nu_{31}\nu_{32} + \nu_{32}\nu_{41} \end{pmatrix}$$
$$= -\frac{2}{det V_{2L}} \begin{pmatrix} {}_{31}\nu_{42} + \nu_{32}\nu_{41} \\ -2\nu_{31}\nu_{32} \\ 2\nu_{41}\nu_{42} \\ -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41} \end{pmatrix}.$$

and

(4.39)

$$Im\left[\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\right] = -\frac{i}{det\ V_{2L}} \nu_A^T \begin{pmatrix} \nu_{31}\nu_{42} + \nu_{32}\nu_{41} & -2\nu_{31}\nu_{32} \\ 2\nu_{41}\nu_{42} & -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41} \end{pmatrix} \nu_M$$

$$= -\frac{i}{det\ V_{2L}} (-\nu_4^*\ \nu_3^*) \begin{pmatrix} \nu_{31}\nu_{42} + \nu_{32}\nu_{41} & -2\nu_{31}\nu_{32} \\ 2\nu_{41}\nu_{42} & -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41}) \end{pmatrix} \begin{pmatrix} \nu_3 \\ \\ \nu_4 \end{pmatrix}.$$

Expressing the complex components as two real components,

(4.40)

$$Im\left[\bar{\nu}_M \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_A\right]$$

$$= -\frac{i}{det\ V_{2L}} \begin{pmatrix} -\nu_{41} & \nu_{31} \\ \nu_{42} & -\nu_{32} \end{pmatrix} \begin{pmatrix} \nu_{31}\nu_{42} + \nu_{32}\nu_{41} & -2\nu_{31}\nu_{32} \\ 2\nu_{41}\nu_{42} & -(\nu_{31}\nu_{42} + \nu_{32}\nu_{41} \end{pmatrix} \begin{pmatrix} \nu_{31} & \nu_{32} \\ \nu_{41} & \nu_{42} \end{pmatrix}$$

$$= -\frac{i}{det\ V_{2L}} \begin{pmatrix} -\nu_{41} & \nu_{31} \\ \nu_{42} & -\nu_{32} \end{pmatrix} det(V_{2L}) \begin{pmatrix} \nu_{31} & -\nu_{32} \\ \nu_{41} & -\nu_{42} \end{pmatrix}$$

$$= -i \begin{pmatrix} 0 & -det\ V_{2L} \\ det\ V_{2L} & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} det\ V_{2L}$$

which is consistent with the evaluation of $\bar{\nu}_A \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_M$ since the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the 2×2 matrix representation of the imaginary unit i. Then

(4.41)
$$\bar{\nu}_A \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix} \nu_M = i\ det\ V_{2L}$$

and

(4.42)
$$m\bar{\nu}\nu = 2m\ Re[\kappa_1^*\kappa_2(\bar{\nu}_M\nu_A + 2i\ det\ V_{2L})].$$

Setting $m\bar{\nu}_M\nu_A$ equal to zero,

(4.43)
$$m\bar{\nu}\nu = 4m\ Re\left[\kappa_1^*\kappa_2\ i\ det\ V_{2L}\right].$$

Since

(4.44)
$$i(\nu_4^*\ -\nu_3^*) \begin{pmatrix} \nu_3 \\ \\ \nu_4 \end{pmatrix} = 2(\nu_{31}\nu_{42} - \nu_{32}\nu_{41}),$$

$$det\ V_{2L} = \frac{i}{2}(\nu_4^* \ \ -\nu_3^*)\begin{pmatrix} \nu_3 \\ \\ \nu_4 \end{pmatrix} \tag{4.45}$$

By the definition of ν_L,

$$\bar{\nu}_L = \nu_L^\dagger \gamma^0 = (\nu_3^*\ \nu_4^*\ 0\ 0) \tag{4.46}$$

and

$$\bar{\nu}_L C \gamma_0 \nu_L = (\nu_3^*\ \nu_4^*\ 0\ 0)\begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}\begin{pmatrix} \nu_3 \\ \\ \nu_4 \\ \\ 0 \\ \\ 0 \end{pmatrix} = (\nu_4^* \ \ -\nu_3^*)\begin{pmatrix} \nu_3 \\ \\ \nu_4 \end{pmatrix}. \tag{4.47}$$

It follows that

$$det\ V_{2L} = \frac{i}{2}\bar{\nu}_L C \gamma^0 \nu_L \tag{4.48}$$

when $\nu_L = \frac{1}{2}(1-\gamma_5)\nu$, $\nu_R = \frac{1}{2}(1+\gamma_5)\nu$, $\nu_1 = \nu_3$, $\nu_2 = \nu_4$, and

$$m\bar{\nu}\nu = 2m\ Re[-\kappa_1^*\kappa_2\bar{\nu}_L C \nu_R]. \tag{4.49}$$

Equivalently,

$$(\nu_4^* \ -\nu_3^*)\begin{pmatrix} \nu_3 \\ \\ \nu_4 \end{pmatrix} = \nu_M^T \frac{1}{2}(1-\gamma_5)\frac{1}{2}(1-\gamma_5)\gamma_0\nu_A = \bar{\nu}_M C^{-1}\frac{1}{2}(1+\gamma_5)\gamma^0\nu_A \tag{4.50}$$

and

$$m\bar{\nu}\nu \to 2m\ Re\left[\kappa_1^*\kappa_2\bar{\nu}_M C \cdot \frac{1}{2}(1+\gamma_5)\gamma^0\nu_A\right] \tag{4.51}$$

Now consider

$$2m\ Re[\kappa_1^*\kappa_2\bar{\nu}_L^{(8)} C^2 \nu_R^{(8)}], \tag{4.52}$$

where ν_L and ν_R are chiral projections in an 8-component spinor. The above process of deriving the mass term, and then transforming it to a form such as that given in Eq.(4.51), results in

$$2^{n-2} m\ Re[\kappa_1^*\kappa_2\bar{\nu}_L^{(4(n-1))} C^{n-1} \nu_R^{(4(n-1))}] \tag{4.53}$$

after the $(n-3)^{th}$ iteration, where ν_L^{4k} and ν_R^{4k} are embedded in spinors with 2^{n_k} components with $2^{n_k-1} < 4k \leq 2^{n_k}$ components, if the initial bilinears in the expansion of $\bar{\nu}_L^{4(\ell-1)}\nu_R^{(4(\ell-1)}$ vanish, similarly to $\bar{\nu}_M\nu_A$, for $2 \leq \ell \leq n-1$. The mass may be left in the form

$$2^{n-1}m\ Re[\kappa_1^*\kappa_2\bar{\nu}_M^{(4n)}C^n\Gamma^{(n)}\nu_A^{(4n)}]_{(0)} \tag{4.54}$$

after the $(n-2)^{nd}$ iteration, where $\nu_M^{(4n)}$,$\nu_A^{(4n)}$ are spinors with $4n$ non-zero components, $\Gamma^{(n)}$ is a product of gamma, generalized charge conjugation and transformation matrices, and $[\bar{\nu}_M^{(4n)}\Gamma^{(n)}\nu_A^{(4n)}]_{(0)}$ denotes those terms proportional to a scalar bilinear.

5. The Maximal Iteration of the Transformation of the Scalar Bilinear

The transformation from the bilinear $\bar{\nu}_L\nu_R$ to $\bar{\nu}_M\nu_A + 2\bar{\nu}_M\begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}\nu_A$ reflects the embedding of two-component spinors into four-component spinors and SL$(2,\mathbb{R})$ into SL$(4;\mathbb{R})$ and SL$(2;\mathbb{C})$. Iterations of this transformation similarly would require the embedding of spinors with $4k$ components into spinors with 2^{n_k} components, where $2^{n_k-1} < 4k \leq 2^{n_k}$, characteristic of spinors in $2n_k$ and $2n_k+1$ dimensions.

The spinor space of the standard model $\oplus_{i=1}^3(\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O})_i$ is a direct sum over the three generations of fermions. When a single generation consists of a lepton, lepton neutrino of both chiralities and two quarks with three different colours. amd two chiral lepton neutrinos can be combined in a four-component spinor, the number of particle and antiparticles would equal 16, The automorphism group of the spinor space is the wreath product $\mathrm{G}_2 \times \mathrm{SU}(2) \times \mathrm{SU}(1)\ wr\ S_3$, where S_3 is the symmetric permutation group. The fundamental representation of G_2 is seven-dimensional, and the theory of the strong interactions has this symmetry at higher energies with the charges of the quarks having seven values. The fermions in one generation in a unified model of the fundamental interactions then would consist of 32 particles and antiparticles.

The expectation value of an operator may be given by quotient a sum of over a complete set of states by the quotient of the sum over states of the expectation value of the identity operator

$$\langle 0|\bar{\nu}\nu|0\rangle = \frac{\sum_\alpha \langle 0|\bar{\nu}|\alpha\rangle\langle\nu|\alpha|0\rangle}{\sum_\alpha\langle 0|\alpha\rangle\langle\alpha|0\rangle} \tag{5.1}$$

The physical states must be identified with the fermions and anti-fermions in each generation. In a unified theory with a $G_2 \times SU(2) \times U(1)$ symmetry.

The iteration of the process described in §4 generates a spinor with $4n$ components, which represents all of the quantum states in place of the particle and antiparticles in the fermion multiplets simultaneously when $n = 32$. This spinor, related to the neutrino that will be designated to be ν_1, can be decomposed into a tensor product of four-component spinors. Then the above method is sufficient to generate from ν_1 an entire set of quantum states, $\{\nu_1, \nu_2, ..., \nu_{16}\}$ in place of the particles in one fermion generation and substitute $\{\bar{\nu}_1, \bar{\nu}_2, ..., \bar{\nu}_{16}\}$ for the antiparticles. Setting

(5.2)

$$\begin{aligned} \{|\alpha\rangle\} = \Big[\langle 0\rangle_1 + |\nu\rangle_1 + \ ... \ + |\nu\nu...\nu\rangle_1\Big] \otimes \Big[|0\rangle_2 + |\nu\rangle_2 + \ ... \ + \ |\nu\nu...\nu\rangle_2\Big] \\ ... \otimes \Big[|0\rangle_{16} + |\nu\rangle_{16} + \ ... \ + |\nu\nu...\nu\rangle_{16}\Big] \\ \otimes \Big[|\bar{0}\rangle_1 + |\bar{\nu}\rangle_1 + \ ... \ + |\bar{\nu}\bar{\nu}...\bar{\nu}\rangle_1\Big] \\ ... \otimes \Big[|\bar{0}\rangle_{16} + |\bar{\nu}\rangle_{16} + \ .... \ + |\bar{\nu}\bar{\nu}...\bar{\nu}\rangle_{16}\Big] ||_{(32)} \end{aligned}$$

where the states are restricted to have a maximum of 32 entries. Decomposing this spinor into a tensor product of four-component spinors, the numerator equals

(5.3)

$$\begin{aligned} \sum_\alpha \langle 0|\bar{\nu}_1...\bar{\nu}_{16}|\alpha\rangle\langle\alpha|\nu_1...\nu_{16}...\rangle = \langle 0|\bar{\nu}_1...\bar{\nu}_{16}\nu_1...\nu_{16}|\nu_1...\nu_{16}\bar{\nu}_1...\bar{\nu}_{16}\rangle \\ \langle\nu_1...\nu_{16}\bar{\nu}_1...\bar{\nu}_{16}|\nu_1...\nu_{16}\bar{\nu}_1...\bar{\nu}_{16}|0\rangle \end{aligned}$$

because only the single quantum state consisting of all fermions in the generation will produce a nonvanishing expectation value. The denominator is

$$\begin{aligned} \sum_\alpha \langle 0|\alpha\rangle\langle\alpha|0\rangle &= \langle 0|0\rangle_1 \, {}_1\langle 0|0\rangle + ... + \langle 0|0\rangle_{16} \, {}_{16}\langle 0|0\rangle \\ &\quad + \langle 0|\bar{0}\rangle_1 \, {}_1\langle\bar{0}|0\rangle + ... + \langle 0|\bar{0}\rangle_{16} \, {}_{16}\langle\bar{0}|0\rangle \\ &= 32. \end{aligned} \tag{5.4}$$

The process described in §4 cannot be continued beyond the 30^{th} iteration because the sum is evaluated over the 32 quantum states in one generation. Then, the vacuum expectation value of the mass term would be

$$\frac{1}{32} 2^{31} m_\nu \langle 0|Re[\kappa_1^* \kappa_2 \bar{\nu}_M^{(4(32))} C^{32} |\nu^{(4(32))}\rangle\langle\nu^{(4(32))}|\Gamma^{(32)} \nu_A^{(4(32))}]_{(0)}|0\rangle. \tag{5.5}$$

Since $C^4 = \mathbb{I}$, the effective mass equals approximately

$$\frac{1}{32} 2^{31} m_\nu = 2^{26} m_\nu. \tag{5.6}$$

The mass of the electron [14] is experimentally measured to be

$$m_e = 0.51099895000 \pm 0.00000000015 \ MeV/c^2. \tag{5.7}$$

If the electron is the final state of the algorithm, the effective mass of the electron neutrino predicted by this algorithm is

$$m_{\nu_e} \approx 0.0076144777238368 \pm 0.000000000002235 \ eV/c^2, \tag{5.8}$$

which coincides with the experimental value.

This method also will provide a connection between mass terms of leptons and neutrinos in the second and third generations. Extra structure in the geometrical models of quarks occurs in these generations. The description of leptons, however, would remain uniform in the absence of this structure. Furthermore, Koide's relation is valid both for leptons [10]

$$m_e + m_\mu +_\tau = \frac{2}{3} \left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau} \right)^2 \tag{5.9}$$

and neutrinos [8][11][15]

$$m_{_e} + m_{\nu_\mu} + m_{\nu_\tau} = \frac{2}{3} \left(\sqrt{m_{\nu_e}} + \sqrt{m_{\nu_\mu}} + \sqrt{m_{n_\tau}} \right)^2 . \tag{5.10}$$

These numerical equalities are compatible when the proportionality between the masses of the lepton and neutrino continues to hold for the second and third generations. Given the masses of the muon and the τ lepton [14]

$$\begin{aligned} m_\mu &= 105.6583755 \pm 0.0000023 \ MeV/c^2 \\ m_\tau &= 1776.82 \pm 0.16 \ MeV/c^2, \end{aligned} \tag{5.11}$$

the neutrino masses then would be predicted to be

$$\begin{aligned} m_{\nu_\mu} &= 1.574432484 \pm 0.000000034 \ eV/c^2 \\ m_{\nu_\tau} &= 26.47668 \pm 0.00238 \ eV/c^2. \end{aligned} \tag{5.12}$$

6. Derivation of the Modified Nonrelativistic Wave Equation

The Schrödinger equation may be derived from the Hamilton-Jacobi equation [13] and the path integral [6]. Given the Hamilton principal function or action

$$S(\vec{q},t) = \int L dt, \tag{6.1}$$

the Hamilton-Jacobi equation is

$$-\frac{\partial S}{\partial t} = H\left(\vec{q}, \frac{\partial S}{\partial \vec{q}}, dt\right) \tag{6.2}$$

where $\frac{\partial S}{\partial \vec{q}} = \vec{p}$. The Hamiltonian of a nonrelativistic particle of mass m in a potential $V(q)$ is

$$H(\vec{q},\vec{p},t) = \frac{1}{2m}|\vec{p}|^2 + V(\vec{q}) \tag{6.3}$$

and

$$H\left(q, \frac{\partial S}{\partial \vec{q}}, t\right) = \frac{1}{2m}\left(\frac{\partial^2 S}{\partial (q^1)^2} + \frac{\partial^2 S}{\partial (q^2)^2} + \frac{\partial^2 S}{\partial (q^3)^2}\right) + V(\vec{q}). \tag{6.4}$$

Then the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial^2 S}{\partial (q^1)^2} + \frac{\partial^2 S}{\partial (q^2)^2} + \frac{\partial^2 S}{\partial (q^3)^2}\right) + V(\vec{q}) = -\frac{\partial S}{\partial t}. \tag{6.5}$$

If $\psi(\vec{q},t) = e^{\frac{i}{\hbar}S(\vec{q},t)}$,

$$\begin{aligned}\frac{\partial S}{\partial q^1} &= -\frac{i\hbar}{\psi}\frac{\partial \psi}{\partial q^1}\\ \frac{\partial S}{\partial q^2} &= -\frac{i\hbar}{\psi}\frac{\partial \psi}{\partial q^2}\\ \frac{\partial S}{\partial q^3} &= -\frac{i\hbar}{\psi}\frac{\partial \psi}{\partial q^3},\\ \frac{\partial S}{\partial t} &= -\frac{i\hbar}{\psi}\frac{\partial \psi}{\partial t}.\end{aligned} \tag{6.6}$$

and

$$\frac{1}{2m}\left(-\frac{\hbar^2}{\psi}\right)\nabla^2\psi + V(\vec{q}) = \frac{i\hbar}{\psi}\frac{\partial \psi}{\partial t} \tag{6.7}$$

At points where the wavefunction does not vanish, such as the interior of a potential well, this equation can be multiplied by ψ to give

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{q})\psi = i\hbar\frac{\partial \psi}{\partial t}. \tag{6.8}$$

To modify the wave equation, it is necessary to alter the Hamiltonian. Multiplying the kinetic term by 3 in Eq.(5.4),

$$H\left(q, \frac{\partial^2 S}{\partial q}, t\right) = \frac{3}{2m}\left(\frac{\partial S}{\partial (q^1)^2} + \frac{\partial^2 S}{\partial (q^2)^2} + \frac{\partial^2 S}{\partial (q^3)^2}\right) + V(\vec{q}) \tag{6.9}$$

for particles that can be described by Majorana spinors. The Hamilton-Jacobi equation becomes

$$\frac{3}{2m}\left(\frac{\partial^2 S}{\partial (q^1)^2} + \frac{\partial^2 S}{\partial (q^2)^2} + \frac{\partial^2 S}{\partial (q^3)^2}\right) + V(\vec{q}) = -\frac{\partial S}{\partial t}. \tag{6.10}$$

By Eq.(6.6),

$$-\frac{3\hbar^2}{2m}\nabla^2\psi + V(\vec{q})\psi = i\hbar\frac{\partial\psi}{\{}\partial t. \tag{6.11}$$

If ψ is independent of x^1 and x^3, and $\psi(\vec{q}, t) = \psi_0(\vec{q})e^{-\frac{i}{\hbar}Et}$,

$$-\frac{3\hbar^2}{2m}\partial_2^2\psi + V(\vec{q})\psi = E\psi \tag{6.12}$$

which is equivalent to Eq.(2.5).

The path integral over an infinitesimal time integral gives the amplitude [9] from t to $t+\varepsilon$,

$$\psi(y; t+\varepsilon) = \int_{-\infty}^{\infty} Dx\ \psi(x;t)\int_{x(t)}^{x(t+\varepsilon)} e^{iS(x,\dot{x},t)}D[x(t)]. \tag{6.13}$$

Over an infinitesimal time interval, the action of a nonrelativistic particle of mass m in a potential $V(x)$ is

$$\int_{x(t)}^{x(t+\varepsilon)} L(x,\dot{x},t)dt \approx \left(\frac{1}{2m}\dot{x}^2 - V(x)\right)\varepsilon. \tag{6.14}$$

By the Baker-Campbell-Hausdorff formula,

$$e^{\left(i\left(\frac{1}{2m}\dot{x}^2 - V(x)\right)\varepsilon + \frac{1}{4m}[\dot{x}^2, V(x)]\varepsilon^2 + O(\varepsilon^3)\right)} = e^{\frac{i}{2m}\dot{x}^2\varepsilon}e^{-iV(x)\varepsilon}. \tag{6.15}$$

Then

$$\psi(y; t+\varepsilon) = \int \psi(x;t)e^{-iV(x)\varepsilon}e^{\frac{i}{2m}\frac{(x-y)^2}{\varepsilon}}dx + O(\varepsilon^2) \tag{6.16}$$

Since $e^{\frac{i}{2m}\frac{(x-y)^2}{\varepsilon}} \to 0$ for $x \neq y$ and $e^{\frac{i}{2m}\frac{(x-y)^2}{\varepsilon}} \to 1$ for $x \to y$, it is sufficient to regard $\frac{(x-y)^2}{\varepsilon}$ as an expansion $\frac{1}{\varepsilon}f(x,y) \approx \frac{1}{\varepsilon}\left[f(y,y) + \varepsilon\frac{\partial f}{\partial x}\right|_{y\approx x}(x-y) + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}\Big|_{y\approx x}(x-$

$y)^2 + ...\Big]$, where $f(x,y)\Big|_{y\approx x} \approx 0$, $\frac{\partial f(x,y)}{\partial} x\Big|_{y\approx x} \approx 0$ and $\frac{\partial^2 f}{\partial x}\Big|_{y\approx x} \approx 2$. Setting $y - x = \varepsilon$, integration by parts and division by ε by parts yields

$$\frac{\partial\psi}{\partial t} = \frac{i}{2m}\frac{\partial^2\psi}{\partial x^2} - V(x)\psi \tag{6.17}$$

in the limit $\varepsilon \to 0$. Replacing the Lagrangian by $\frac{3}{2m}\dot{x}^2 - V(x)$, the path integral over the infinitesimal time slice $[t, t+\varepsilon]$ produces the equation

$$-\frac{3}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\frac{\partial\psi(x,t)}{\partial t} \tag{6.18}$$

which is the time-dependent version of the modified nonrelativistic wave equation in §2 in the units with $\hbar = 1$.

7. Conclusion

The generalization of the momentum operator to introduce a real component of the quantum commutation relations requires a matrix term for hermiticity. The resulting nonrelativistic equations are matrix equations for a four-component spinor. It may be reduced to a differential equation for a real-valued Majorana or anti-Majorana spinors. The Schrödinger equation with an effective mass reduced by a factor of 3 is derived for the real-valued Majorana spinors. The second class of real anti-Majorana spinors would solve the standard Schrödinger equation.

These conditions on the spinors may be combined with the Dirac conjugate to prove the vanishing of a set of bilinears, including $\bar{\psi}\psi$ for both classes. Furthermore, it can be checked that an anti-particle is defined by the equivalent of an anti-Majorana condition when the particle is represented by a Majorana spinor. The bilinear representing the current of this anti-Majorana anti-fermion, with a Majorana fermion, may be demonstrated to be zero. This result provides a theoretical explanation of the zero-energy state consisting of negative-energy electron states being filled and essentially no positron current.

Given this discussion of the positron current, and a theoretical prediction for the reaction $e^- \to W^-\nu_e$ in cosmic rays, the electron can be described as a linear combination of Majorana spinor and an anti-Majorana spinor, where the coefficient of the first field being much larger. The neutrino is a nearly massless, chiral spin-$\frac{1}{2}$ field. Since the electron and the chiral neutrino form an SU(2) in the electroweak model, the existence of transformations from Weyl spinors to Majorana spinors is investigated. The matrix may be mapped bijectively to a subset of an $SL(2;\mathbb{R})$ subgroup of $SL(2;\mathbb{C})$, the spin covering of the Lorentz group, and it is separate from an isospin rotation. Both the neutrino and the electron would have a nonvanishing mass by the existence of this transformation.

The mass term of the neutrino is transformed into a bilinear consisting of a Majorana spinor, a product of matrices and anti-Majorana spinors. When the $\mathbb{Z}_2$ transformation is included, the product equals the negative of the identity. If the set of transformation matrices is given by multiplication with γ_5, the product in the bilinear may be equated to the sum of the identity and a nilpotent matrix. The identity yields a scalar bilinear $\bar{\nu}_M \nu_A$ that vanishes. The second matrix gives rise to $2i\ det\ V_{2L}$, where the matrix V_{2L} is composed of the real components of ν_L. This determinant may be reformulated in terms of $\bar{\nu}_L$ and ν_R, the initial chiral spinors. Then, mass term of a new quantum state can be defined by projecting a product charge conjugation and γ matrices to the identity, and replacing the spinors in the bilinear by $\bar{n}u_L^{(4)}$ and $\nu^{(R)}$. These four-component spinors would be the equivalent of the two-component spinors upon the embedding into a general eight-component spinor. The numerical coefficient of the bilinear increases by a factor of 2.

This algorithm then generates a sequence of mass terms of quantum states that stops after a finite number of iterations. The number of iterations can be related to the number of components of spinors representing fermions in a single generations as a result of a completeness relation for the expectation value of an operator. Since the number of spinor components of only particles in one fermion generation is 32, the number of iterations of the initial transformation is 30, and normalization of the expectation value requires a quotient by 32. Then the effective mass of the final state is increased from the neutrino mass by a factor of $\frac{1}{32}2^{31} = 2^{26}$. The final quantum state in the sequence would be the lepton. Recent experimental measurments of the mass of the electron neutrino mass coincide with $\frac{1}{2^{26}}$ of the electron mass. The proportionality between the lepton and neutrino mass in the second and third generations is consistent with the Koide relation. It yields theoretical predictions for m_{ν_μ} and m_{ν_τ} and verifies the direct sum structure of the spinor space of the standard model.

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